THE ROLE OF TOPOLOGY IN DECISION THEORY

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Outline

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Historical background

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The marginalist period: 1838-1947

Cournot (theory of the firm and single market equilibrium, 1838), Walras (theory of the consumer and general equilibrium, 1874), Edgeworth (exchange economy and contract curve, 1881), Marshall (demand theory, 1890), Pareto (general equilibrium and optimal resource allocation, 1896), Hicks (stability of equilibrium and barganing, 1946),...,etc.

Historical background (continued)

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The set-theoretic/linear models period: 1848-1960

Arrow (social choice theory, 1951), Arrow-Debreu (general equilibrium, 1954), McKenzie (general equilibrium, 1954), von Neumann and Morgenstern (game theory, 1947), Nash (game theory, 1950), Dantzig (linear programming, 1949),...,etc.

Historical background (continued)

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The period of integration: 1961-1980

Debreu (regular economies, 1970), Aumann (large economies, 1964), Sen (social choice, 1970), Koopmans (optimal growth theory, 1965), Smale (global analysis, 1976),...,etc.

Historical background (continued)

The period of integration: 1961-1980

Debreu (regular economies, 1970), Aumann (large economies, 1964), Sen (social choice, 1970), Koopmans (optimal growth theory, 1965), Smale (global analysis, 1976),...,etc.

The period of spread: 1981-present

Decision theory (foundations, utility theory, risk, uncertainty,...., social choice), extensions of the Arrow-Debreu model (incomplete markets, financial markets, infinite dimensional spaces,...),...,etc.

Forewords

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- *Utility theory*: A term that formerly comes from economics and related disciplines (also used in ethics, philosophy, psychology,..) and refers to a measure of the individual (collective) welfare (Jevons, Edgeworth, Pareto, Wold,...).

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- Confusion between preference and utility: "if a set of items is strongly ordered, it is such that each item has a place of its own in the order; it could, in principle, be given a number" (Hicks (1956, p.19)).

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- Contributions to utility theory come from many disparate sources: pure maths (including operations research and statistics), psychology (measurement theory), economics,....
- *Social choice theory*: Is it possible to aggregate individual preferences into a social one ?

Objectives

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Goals:

- Presenting in a unified framework both classical and recent results in utility theory.
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- Presenting in a unified framework both classical and recent results in utility theory.
- Introducing to social choice theory: combinatorial and topological models.

Outline

The representation problem

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Theorem (Cantor, 1895)

Let $(X, \precsim)$ be a totally ordered set that is unbordered, dense, and denumerable. Then there exists an order monomorphism from X onto $\mathbb{Q}$.

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Let X be, $P = \{\precsim \subseteq X \times X;\ \precsim \text{ transitive and total}\}$ and $S = \{(a,b) \in \mathbb{R}^2;\ a \leq b\}$

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Question:

$\precsim\, \in P, \exists\ u : X \longrightarrow \mathbb{R}\ ;\ (x, y) \in\, \precsim\, \Longleftrightarrow (u(x), u(y)) \in S$?

The representation problem (continued)

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Notation. $P_R = \{\precsim \in P \; ; \precsim \text{ is representable}\}$

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Definition

$\precsim \in P$ is *perfectly separable* if $\exists\ D \subseteq X$, D countable in such a way that for every $x, y \in X$, with $(y, x) \notin \precsim$, there is $d \in D$ such that $(x, d), (d, y) \in \precsim$.

Characterizing representability

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Notation. For a given $\precsim \in P$, $\tau_{\precsim}$ denotes the order topology on X.

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Theorem

Let $\precsim \in P$. Then $\precsim \in P_R$ iff $\tau_{\precsim}$ is second countable.

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Two canonical examples:

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1. Lexicographic plane:

$(\mathbb{R}^2, \precsim_L)$:

$(x_1, x_2) \precsim_L (y_1, y_2) \iff x_1 < y_1$, or $x_1 = y_1$ then $x_2 \leq y_2$.

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2. The first uncountable ordinal:

$([0, \omega_1), \precsim)$:

$W \precsim T \iff W,\ T \in [0, \omega_1)$, W is order isomorphic to an ideal of T.

Definitions and questions

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Definition.

A chain $(X, \precsim)$ is said to be:
(i) *planar (or, a Debreu chain)* if it contains a subchain that is order isomorphic to a non-representable subset of the lexicographic plane $(\mathbb{R}^2, \precsim_L)$.
(ii) *long* if it, or its dual, contains a subchain which is order isomorphic to $[0, \omega_1)$.

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(ii) *long* if it, or its dual, contains a subchain which is order isomorphic to $[0, \omega_1)$.

Questions:

(1) Are there any other orders, different from the above two, that will cause problems ?

(2) How many such other orders can we find ?

Solving question (1):

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Definition.

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An uncountable short (i.e., no long) chain that does not contain any uncountable representable subchain is called an *Aronszajn chain*.

Theorem (Beardon et al., 2002)

(i) There exists an Aronszajn chain.
(ii) Every Aronszajn chain is order isomorphic to a subchain of $(\mathbb{Q}_{00}^{\omega_1}(-1,1), \precsim_L)$.

Solving question (2):

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Theorem (Beardon et al., 2002)

Let $(X, \precsim)$ be an arbitrary chain. Then the following assertions are equivalent:
(i) $\precsim \notin P^a_R$.
(ii) $(X, \precsim)$ is long chain, a Debreu chain or an Aronszajn chain.

Solving question (2):

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(ii) $(X, \precsim)$ is long chain, a Debreu chain or an Aronszajn chain.

Remark.

A *Souslin chain* contains a copy of an Aronszajn chain.

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Notation.

- $P_C = \{\precsim \in P\ ; \precsim$ is closed in $X \times X\}$
- $P_{CR} = \{\precsim \in P\ ; \precsim$ admits a continuous utility function$\}$

Characterizing P_{CR}

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Theorem

$P_{CR} = P_C \bigcap P_R$.

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$P_{CR} = P_C \bigcap P_R$.

Corollary (Debreu's open-gap lemma)

If $\tau \equiv \tau_{\precsim}$, for some $\precsim \in P$, then $\precsim \in P_{CR}$ iff $\precsim \in P_R$.

Two cornerstone results

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Theorem (Eilenberg, 1941)

Let (X, τ) be a connected and separable topological space. Then any continuous total preorder defined on X has a continuous utility function.

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Let (X, τ) be a connected and separable topological space. Then any continuous total preorder defined on X has a continuous utility function.

Theorem (Debreu, 1964)

Let (X, τ) be a second countable topological space. Then any continuous total preorder defined on X has a continuous utility function.

CRP: Definition

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Definition

Let (X, τ) be a topological space. Then τ satisfies the *Continuous Representability Property (CRP)* if $P_C = P_{CR}$.

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(ii) If τ is second countable (Debreu, 1964).

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Theorem

Let (X, τ) be a topological space. Then CRP holds true in the following cases:
(i) If τ is connected and separable (Eilenberg, 1941).
(ii) If τ is second countable (Debreu, 1964).
(iii) If τ is path-connected and σ-compact (Monteiro, 1987).

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(iii) If τ is path-connected and σ-compact (Monteiro, 1987).
(iv) If τ is locally connected and separable (Candeal et al., 2004).

Sufficient conditions for CRP

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Let (X, τ) be a topological space. Then CRP holds true in the following cases:

(i) If τ is connected and separable (Eilenberg, 1941).

(ii) If τ is second countable (Debreu, 1964).

(iii) If τ is path-connected and σ-compact (Monteiro, 1987).

(iv) If τ is locally connected and separable (Candeal et al., 2004).

(v) If τ is separably connected and satisfies CCC. In particular, this implies that the weak topology of a Banach space has CRP (Campión et al., 2006).

Some characterizations of CRP

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Theorem

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Theorem

Let $(X, \precsim)$ be a totally preordered space. Then $\tau_{\precsim}$ satisfies CRP iff it is second countable.

Some characterizations of CRP (continued)

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Theorem

Let $(V, +, \cdot_{\mathbb{R}}, \tau)$ be a locally convex topological real vector space. Then τ satisfies CRP iff it satisfies CCC.

Characterizing CRP

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Definition

A topology τ on X is said to be *(totally) preorderable* if $\tau \equiv \tau_{\precsim}$, for some $\precsim \in P$.

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A topology τ on X is said to be *(totally) preorderable* if $\tau \equiv \tau_{\precsim}$, for some $\precsim \in P$.

Theorem (Herden and Pallack, 2000 and Campión et al., 2002)

Let (X, τ) be a topological space. The topology τ satisfies CRP if and only if all its preorderable subtopologies are second countable.

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The additive representation problem

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Let $(X, +)$ be a group, P the space of preferences and $S = \{(a, b) \in \mathbb{R}^2;\ a \leq b\}$

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Question:

$\precsim\ \in P$, $\exists\ u: X\longrightarrow\mathbb{R}$, u additive, such that $(x,y)\in\ \precsim\ \Longleftrightarrow (u(x),u(y))\in S$?

The additive representation problem

Let $(X,+)$ be a group, P the space of preferences and $S = \{(a,b) \in \mathbb{R}^2;\ a \leq b\}$

Question:

$\precsim \in P$, $\exists\ u : X \longrightarrow \mathbb{R}$, u additive, such that $(x,y) \in \precsim \iff (u(x), u(y)) \in S$?

Definition

Let $(X,+)$ be a group. A total preorder $\precsim \in P$ is said to be *translation-invariant* if $x \precsim y \Longrightarrow x + z \precsim y + z$ and $z + x \precsim z + y$, $\forall x, y, z \in X$.

Characterizing P_{AR}

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Remark. $P_{AR} \subsetneq P_R \bigcap P_A$.

Characterizing P_{AR}

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$P_{AR} = \{\precsim \in P \,; \precsim$ admits an additive utility function$\}$.
Remark. $P_{AR} \subsetneq P_R \bigcap P_A$.

Theorem

Let $(X, +)$ be a group and let $\precsim \in P$. Then the following assertions are equivalent:
(i) $\precsim \in P_{AR}$.
(ii) $\precsim \in P_A$ and there is a base of the origin for $\tau_{\precsim}$, say V_e, such that, for each $B \in V_e$, $\bigcup_{n \in \mathbb{N}} nB = X$.

The continuous additive representation problem

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Let $(X, +)$ be a group, τ a topology on X, P the space of preferences and $S = \{(a, b) \in \mathbb{R}^2;\ a \leq b\}$

Question:

$\precsim \in P$, $\exists\ u : X \longrightarrow \mathbb{R}$, u continuous and additive, such that $(x, y) \in \precsim \iff (u(x), u(y)) \in S$?

Notation. $P_{CAR} = \{\precsim \in P\ ; \precsim$ admits a continuous and additive utility function$\}$.

Characterizing P_{CAR}

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Theorem (De Miguel et al., 1998)

$P_{CAR} = P_C \bigcap P_{AR}$.

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$P_{CAR} = P_C \bigcap P_{AR}$.

Corollary (algebraic version of the open-gap lemma for groups)

If $\tau \equiv \tau_{\precsim}$, for some $\precsim \in P$, then $\precsim \in P_{CAR}$ iff $\precsim \in P_{AR}$.

The continuous algebraic representability property

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Definition

Let $(X, +)$ be a group and τ a topology on X. Then τ satisfies the *Continuous Algebraic Representability Property (CARP)* if $P_{CAR} = P_C \bigcap P_A$.

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Let $(X, +)$ be a group and τ a topology on X. Then τ satisfies the *Continuous Algebraic Representability Property (CARP)* if $P_{CAR} = P_C \bigcap P_A$.

Theorem

Let $(X, +)$ be a group and τ a topology on X. Then the following assertions are equivalent:
(i) τ has CARP
(ii) For each $\precsim \in P_C \bigcap P_A$, there is an absorbing basis of the origin for $\tau_{\precsim}$.

A sufficient condition for CARP

Corollary

Let $(X, +)$ be a group and τ a connected topology on X. Then τ satisfies CARP.

The linear representation problem

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Let $(X, +, \cdot_{\mathbb{R}})$ be a real vector space, P the space of preferences and $S = \{(a, b) \in \mathbb{R}^2;\ a \leq b\}$

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Question:

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The linear representation problem

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Question:

$\precsim \in P$, $\exists\ u : X \longrightarrow \mathbb{R}$, u linear, such that $(x, y) \in \precsim \iff (u(x), u(y)) \in S$?

Definition

Let $(X, +, \cdot_{\mathbb{R}})$ be a real vector space. A total preorder $\precsim \in P$ is said to be *homothetic* if $x \precsim y \Longrightarrow \lambda x \precsim \lambda y$, $\forall\ x, y \in X$, $\lambda \geq 0$.

Characterizing P_{LR}

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Notation. $P_H = \{\precsim \in P\ ; \precsim \text{ is homothetic}\}$.
$P_L = P_A \bigcap P_H$.
$P_{LR} = \{\precsim \in P\ ; \precsim \text{ admits a linear utility function}\}$.

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Theorem

Let $(X, +, \cdot_{\mathbb{R}})$ be a real vector space and let $\precsim \in P$. Then the following assertions are equivalent:
(i) $\precsim \in P_{LR}$.
(ii) $\precsim \in P_L$ and $(X, +, \cdot_{\mathbb{R}}, \tau_{\precsim})$ is a topological real vector space.

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(i) $\precsim \in P_{LR}$.
(ii) $\precsim \in P_L$ and $(X, +, \cdot_{\mathbb{R}}, \tau_{\precsim})$ is a topological real vector space.

Consequence. $P_{LR} = P_L \bigcap P_R$.

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Question:

$\precsim \in P$, $\exists\ u : X \longrightarrow \mathbb{R}$, u continuous and linear, such that $(x, y) \in \precsim \iff (u(x), u(y)) \in S$?

Characterizing P_{CLR}

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Notation. Let $(X, +, \cdot_{\mathbb{R}})$ be a real vector space equipped with a topology τ. $P_{CLR} = \{\precsim \in P \, ; \precsim$ admits a linear and continuous utility function$\}$.

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Theorem

$P_{CLR} = P_C \bigcap P_{LR}$.

Corollary (algebraic version of the open-gap lemma for real vector spaces)

If $\tau \equiv \tau_{\precsim}$, for some $\precsim \in P$, then $\precsim \in P_{CLR}$ iff $\precsim \in P_{LR}$.

An important consequence

Corollary

The only topological totally ordered real vector space is $\mathbb{R}$ or its dual.

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The continuous algebraic representability property

Definition

Let $(X, +, \cdot_{\mathbb{R}})$ be a real vector space equipped with a topology τ. Then τ satisfies the *Continuous Algebraic Representability Property (CARP)* if $P_{CLR} = P_C \bigcap P_L$.

Theorem

Let $(X, +, \cdot_{\mathbb{R}})$ be a real vector space equipped with a topology τ. Then the following assertions are equivalent:

(i) τ has CARP

(ii) For each $\precsim \in P_C \bigcap P_L$, $(X, +, \cdot_{\mathbb{R}}, \tau_{\precsim})$ is a topological real vector space.

A sufficient condition for CARP

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Corollary

Let $(X, +, \cdot_{\mathbb{R}})$ be a real vector space equipped with a connected topology τ. Then τ satisfies CARP. In particular, if $(X, +, \cdot_{\mathbb{R}}, \tau)$ is a topological real vector space, then τ satisfies CARP.

A sufficient condition for CARP

Corollary

Let $(X, +, \cdot_{\mathbb{R}})$ be a real vector space equipped with a connected topology τ. Then τ satisfies CARP. In particular, if $(X, +, \cdot_{\mathbb{R}}, \tau)$ is a topological real vector space, then τ satisfies CARP.

Remark

The analogous of the last statement of the corollary is not longer for groups.

Outline

Arrow’s model

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Definition

A social welfare functional $F : P^n \longrightarrow P$ satisfies:

- *Pareto* condition if, for any pair of alternatives $x, y \in X$ and any preference profile $(\precsim_j) \in P^n$, we have that $x \precsim_j y$ for all $j \in N$ implies $xF(\precsim_j)y$, and also that $x \prec_j y$ for all j implies $xF(\precsim_j)_s y$.
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- the *independence of irrelevant alternatives (IIA)* condition if, for any pair of alternatives $x, y \in X$ and any pair of preference profiles $(\precsim_j)$, $(\precsim'_j) \in P^n$ with the property that, for every j, $x \precsim_j y \iff x \precsim'_j y$ and $y \precsim_j x \iff y \precsim'_j x$, we have that $xF(\precsim_j)y \iff xF(\precsim'_j)y$ and $yF(\precsim_j)x \iff yF(\precsim'_j)x$.

Arrow's model (continued)

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Theorem (Arrow, 1951)

Suppose that X contains at least three elements and let $F : P^n \to P$ be a social welfare functional that satisfies Pareto and IIA. Then there is an individual, say i, such that for every profile $(\precsim_j)$ and every $x, y \in X$ it holds that $x \prec_i y \Rightarrow xF(\precsim_j)_s y$.

Arrow's model (continued)

Theorem (Arrow, 1951)

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Remark

Suppose that X contains at least three elements and let $F : P_a^n \rightarrow P$ be a social welfare functional that satisfies Pareto and IIA. Then there is an individual, say i, such that $F(\precsim_j) = \precsim_i$, for every profile $(\precsim_j) \in P_a^n$.

Sketch of proof

Sketch of proof

Definition

A coalition $A \subseteq N = \{1, \ldots, n\}$ is said to be *decisive* if for any pair $x, y \in X$, any profile $(\precsim_j) \in P^n$ such that $x \prec_j y$, for every $j \in A$, then $xF(\precsim_j)_s y$.

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Fact (Fishburn, 1970 and Hansson, 1972)

Under the hypotheses of Arrow's theorem the set of decisive coalitions is an *ultrafilter* on N.

Chichilnisky's model

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Definition

Let (P, τ) be a topological space. A social choice rule $F : P^n \longrightarrow P$ is a map which satisfies:

- *anonymity*, i.e., if the two profiles (p_j), $(q_j) \in P^n$ are the same up to a rearrangement, then $F(p_j) = F(q_j)$.

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- *continuity*, i.e., if F is continuous.

Some impossibility results in topological aggregation

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Suppose that $P = S^1$, linear preferences over the closed disk B^2, and let $n = 2$. Then there is no social choice rule.

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Sketch of proof (Candeal and Induráin, 1994)

If such a social choice rule would exist then the boundary of the Möbius strip would be a continuous deformation retract of the whole strip. But this is not possible.

Some impossibility results in topological aggregation (continued)

Some impossibility results in topological aggregation (continued)

Theorem

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Some impossibility results in topological aggregation (continued)

Theorem

Suppose that $P = S^1$, linear preferences over the closed disk B^2, and let $n \geq 2$. Then there is no social choice rule.

Sketch of proof

Based on Algebraic Topology techniques. If such a rule $F : (S^1)^n \to S^1$ does exist then it can be shown that the fundamental group of S^1, $\pi_1(S^1)$, is Abelian and any element is divisible by n. But this leads to a contradiction since $\pi_1(S^1) = \mathbb{Z}$.

The resolution of the social choice paradox

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Important facts

- (1) Let P be a space of preferences with finitely generated homotopy groups. If a social choice rule does exist, for any $n \in \mathbb{N}$, then each homopoty group of P is trivial.
- (2) *Contractible spaces* have the property that all their homopoty groups are trivial.
- (3) Identify a class of topological spaces (P, τ) for which the converse of (2) holds true.

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The resolution of the social choice paradox (continued)

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Theorem

Suppose that P is connected and homotopic to a *polyhedron*. If a social choice rule exists for any number of individuals, then P is contractible.

The resolution of the social choice paradox (continued)

The resolution of the social choice paradox (continued)

Question

When contractibility is a sufficient condition ?

The resolution of the social choice paradox (continued)

Question

When contractibility is a sufficient condition ?

Theorem

Suppose that P is connected and homeomorphic to a *polyhedron*. Then, a social choice rule exists for any number of individuals iff P is contractible.

The resolution of the social choice paradox (continued)

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When contractibility is a sufficient condition ?

Theorem

Suppose that P is connected and homeomorphic to a *polyhedron*. Then, a social choice rule exists for any number of individuals iff P is contractible.

Theorem (Chichilnisky and Heal, 1983)

Suppose that P is a connected *parafinite CW-complex*. Then a social choice rule exists for any number of individuals iff P is contractible.

Further developments

- (1) Dropping anonymity and replacing unanimity by Pareto condition while keeping continuity leads to the existence of a unique manipulator on spheres.
- (2) Futher remarks and generalizations of Chichilnisky and Heal results can be given by Horvath (2001), Eckmann (2004), Weinberger (2004), Ardanza-Trevijano et al. (2007),. . . , etc.
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Economics does not live by Topology alone

Definition

A social welfare functional $F : P^n \longrightarrow P$ is said to be a *two-sided m serial dictatorship* if there are a natural number $m \in N = \{1, \ldots, n\}$, an injection π of $N(m) = \{1, \ldots, m\}$ into N and a partition $\{A, B\}$ of $\pi(N(m))$ such that, for any pair of alternatives $x, y \in X$ and any preference profile $(\precsim_j) \in P^n$, we have that $xF(\precsim_j)y$ if and only if $x \sim_{\pi(k)} y$ for all $k \in N(m)$ or else, there is $i \in N(m)$ so that $x \sim_{\pi(k)} y$ for all $k < j$ and $x \prec_{\pi(i)} y$ whenever $i \in A$, or $y \prec_{\pi(i)} x$ whenever $i \in B$.

A characterization result in the Arrovian model

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Theorem

Suppose that X contains at least three elements and let $F : P^n \to P$ be a social welfare functional. Then the following assertions are equivalent:
(i) F satisfies PI and IIA,
(ii) F is either trivial, or there is $m \in N$ such that F is a two-sided m serial dictatorship.

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(i) F satisfies PI and IIA,
(ii) F is either trivial, or there is $m \in N$ such that F is a two-sided m serial dictatorship.

Proof

Based on theory of *ordered algebraic systems*.

Outline

Proof 1

Suppose that τ satisfies CCC and let $\precsim$ be a continuous total preorder defined on V. Observe that it is sufficient to show that it is representable. Since (V, τ) is a separably connected topological space, by a result of Campión et al. (2006), there is a continuous order-preserving function $u : V \to L$, where L denotes the long line. Notice that $u(V)$ is an interval of L. Furthermore, there is an ordinal $\alpha_0 \in \omega_1$ that bounds $u(V)$. Indeed, otherwise $u(V)$ exhausts L and then by considering $(V_\alpha)_{\alpha\in\omega_1} = (u^{-1}(\alpha, \alpha+1))_{\alpha\in\omega_1}$, we obtain an uncountable family of pairwise disjoint non-empty open subset of V, which contradicts CCC. So, there is a countable ordinal that bounds $u(V)$ and therefore $u(V)$ can be identified with a subset of the real line. This means that $\precsim$ is representable.

Proof 1 (continued)

For the converse suppose that, by way of contradiction, (V, τ) does not satisfy CCC. Let us then show that there is a continuous total preorder defined on V which is not representable. This will imply that (V, τ) does not satisfy CRP. To that end, let $(p_\beta)_{\beta \in I}$ be a family of seminorms that define the topology τ on V and let $(U_\alpha)_{\alpha \in \omega_1}$ be an uncountable family of pairwise disjoint non-empty open subset of V. Without loss of generality we can assume that, for each α, there is $\epsilon_\alpha > 0$ such that U_α can be chosen of the form $U_\alpha = \{v \in V$; there are $v_\alpha \in V$, $\beta_\alpha \in I$ and $; p_{\beta_\alpha}(v - v_\alpha) < \epsilon_\alpha\}$.

Proof 1 (continued)

Indeed, we can single out an uncountable family of pairwise disjoint non-empty basis open subset of V, say, $(V_\alpha)_{\alpha \in \omega_1}$. Recall that each V_α is of the form $V_\alpha = \{v \in V;$ there are $v_\alpha \in V$, $\beta_1, \ldots, \beta_n \in I$ and $\epsilon_1, \ldots, \epsilon_n > 0$; $p_{\beta_j}(v - v_\alpha) < \epsilon_j$; $j = 1, \ldots, n\}$. Then, by considering for each $\alpha \in \omega_1$, $\epsilon_\alpha =$min$\{\epsilon_j$; $j = 1, \ldots, n\}$ we can assume each U_α to be of the form described above.

Proof 1 (continued)

For each $\alpha \in \omega_1$, consider the (non-trivial) real interval $[0, \epsilon_\alpha]$. Notice that $[0, \epsilon_\alpha] \subset \mathbb{R}$ is order-isomorphic to $[0, \alpha] \subset L$. Denote this order isomorphism by ϕ_α. Define now the function $u : V \to L$ as follows:

$$u(v) = \begin{cases} \phi_\alpha(\epsilon_\alpha - p_\alpha(v - v_\alpha)), & v \in U_\alpha \\ 0 \ , & v \in V \setminus \bigcup_{\alpha \in \omega_1} U_\alpha \end{cases}$$

It remains to prove that u so defined is a continuous function since then by considering the relation on V defined as: $v \precsim_u w$ iff $u(v) \leq u(w)$, $(v, w \in V)$, it is straightforward to see that $\precsim_u$ is a non-representable continuous total preorder.

Proof 1 (continued)

To show the continuity of u let $b \in L$. Then it is sufficient to see that $u^{-1}(L(b))$ and $u^{-1}(G(b))$ are open subsets of V. If $b = 0$ then $u^{-1}(L(b)) = u^{-1}(\emptyset) = \emptyset$ and $u^{-1}(G(b)) = \bigcup_{\alpha \in \omega_1} U_\alpha$ both of which are, obviously, open subsets of V.
Now, if $b \in L \setminus \{0\}$ then for each $\alpha \in \omega_1$ such that $b < \alpha$, let us define the subsets $A_\alpha = \{v \in U_\alpha;\ p_\alpha(v - v_\alpha) > \epsilon_\alpha - \phi_\alpha^{-1}(b)\}$ and $B_\alpha = \{v \in U_\alpha;\ p_\alpha(v - v_\alpha) < \epsilon_\alpha - \phi_\alpha^{-1}(b)\}$. Notice that both A_α and B_α are non-empty open subsets of V. Then,
$u^{-1}(L(b)) = (\bigcup_{b<\alpha} A_\alpha) \bigcup (V \setminus \bigcup_{\alpha \in \omega_1} U_\alpha)$ and
$u^{-1}(G(b)) = \bigcup_{b<\alpha} B_\alpha$, whence open subsets of V too. This concludes the proof.

Proof 2

The "only if" part is obvious. For the converse, let $u : G \to \mathbb{R}$ be an additive order-preserving function for $\precsim$. Let us show that u is also continuous. To that end, recall that if $S \subseteq \mathbb{R}$ is a subgroup of the usual additive group of the reals, then $S = \{0\}$, or there is some $a \in (0, \infty)$; $S = a\mathbb{Z} = \{az; z \in \mathbb{Z}\}$, or S is dense in $\mathbb{R}$ (see, e.g. Choquet (1966, p.56)).
As in the proof of Theorem 4.1 it is sufficient to show that, for every $b \in \mathbb{R}$, $u^{-1}(L(b))$ and $u^{-1}(G(b))$ are open subsets of G. Consider the subgroup of the reals $S = u(G) \subseteq \mathbb{R}$. According to the above result, three cases need to be distinguished. Case (i): If $S = \{0\}$ then the results is obvious because u is constant.

Proof 2 (continued)

Case (ii): If $S = a\mathbb{Z}$, then there are unique $s = u(g), s' = u(g') \in S$ such that $b \in [s, s']$. Thus, $u^{-1}(L(b)) = L(g')$ and $u^{-1}(G(b)) = G(g)$, which are open subsets of G since u is order.preserving and $\precsim$ is continuous.
Case (iii): If S is a dense subgroup of the reals then $u^{-1}(L(b)) = \bigcup_{\{g \in G; u(g) < b\}} L(g)$ and $u^{-1}(G(b)) = \bigcup_{\{g \in G; b < u(g)\}} G(g)$, which are open subsets of G since u is order.preserving and $\precsim$ is continuous. This ends the proof.

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THANKS