Hypo geometry on solvable Lie algebras

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Abstract

The geometry of hypersurfaces in Riemannian manifolds with holonomy contained SU(3) is called *hypo geometry*. It was introduced by Conti and Salamon in [CS]. In order to construct examples of hypo structures, a natural place to look is left-invariant structures on 5-dimensional Lie groups. In five dimensions, **only** 9 **isomorphism classes of nilpotent Lie groups exist, of which exactly six admit a hypo structure** (see [CS]). By Mubarakzyanov's classification [Mub], **there are** 66 **families of solvable Lie groups of dimension** 5, some of which depend on parameters. It was shown in [Da] that precisely 35 out of these 66 families admit an invariant contact structure.

We show obstructions to the existence of hypo structures on Lie groups. In particular, for any Lie algebra $\mathfrak{g}$, the choice of a splitting $\mathfrak{g}^* = V_1 \oplus V_2$, and the vanishing of certain associated cohomology groups, determine a first obstruction. As an application, we obtain a classification of the 5-dimensional solvable Lie algebras carrying a hypo structure, and **we find that only** 21 **of the** 66 **families** have such a structure, so the ratio is considerably less than in the nilpotent case.

1 Hypo geometry

Definition 1.1 *An SU(2)-structure on a 5-manifold consists of a quadruplet $(\alpha,\omega_1,\omega_2,\omega_3)$, of differential forms where α is a 1-form and ω_i are 2-forms, satisfying:*

$$\omega_i\wedge\omega_j=\delta_{i,j}v,\qquad \alpha\wedge v\neq 0$$

for some 4-form v, and

$$X\lrcorner\,\omega_1=Y\lrcorner\,\omega_2=0\Rightarrow\omega_3(X,Y)\geq 0,$$

where $X\lrcorner$ denotes the contraction by X.

The SU(2)-structure determined by (α,ω_i) es called **hypo** *if*

$$d\omega_1=0,\quad d(\alpha\wedge\omega_2)=d(\alpha\wedge\omega_3)=0,$$

Proposition 1.2 *[CS] Let $i: N^5\to M^6$ be an immersion of an oriented 5-manifold into a 6-manifold with an $SU(3)$-structure (ω,ψ_+,ψ_-), and denote by V the unit normal vector field. Then the $SU(3)$-structure on M^6induces an $SU(2)$-structure $(\alpha,\omega_1,\omega_2,\omega_3)$ on N^5 defined by:*

$$\alpha=-V\lrcorner\,\omega,\quad \omega_1=i^*\omega,\quad \omega_2=V\lrcorner\,\psi_-,\quad \omega_3=-V\lrcorner\,\psi_+.$$

Moroever if the $SU(3)$-structure is integrable ($d\omega=d\psi_+=d\psi_-=0$), then the $SU(2)$-structure on N^5 is hypo.

Conversely if N^5 has an $SU(2)$-structure $(\alpha,\omega_1,\omega_2,\omega_3)$, then it induces an $SU(3)$-structure (ω,ψ_+,ψ_-) on $N^5\times\mathbb{R}$, defined by:

$$\omega=\omega_1+\alpha\wedge dt,\quad \psi=\psi_++i\psi_-=(\omega_2+i\omega_3)\wedge(\alpha+idt)$$

Moroever, Conti and Salamon [CS] prove that, in the real analytic case, every hypo structure on N^5 is induced by an immersion of N^5 into a 6-manifold with $SU(3)$-structure.

Local Expression. We say that the coframe $e^1,\dots,e^5$ is *adapted* to the hypo structure if is a local basis $\{e^1,\dots,e^5\}$ of 1-forms on N^5 such that

$$\alpha=e^5,\quad \omega^1=e^{12}+e^{34},\quad \omega^2=e^{14}+e^{32},\quad \omega^{34}=e^{13}+e^{24}.$$

2 Obstructions for existence of hypo structures.

We introduce some obstructions to the existence of a hypo structure on a Lie algebra.

Let $\mathfrak{g}$ be a 5-dimensional Lie algebra, and denote by d the Chevalley-Eilenberg differential on the dual $\mathfrak{g}^*$. A **coherent splitting** of $\mathfrak{g}$ is a splitting $\mathfrak{g}^*=V_1\oplus V_2$, where V_1 and V_2 are vector spaces,

$$\mathbf{dim V_1=2}\quad\text{and}\quad \mathbf{d(V_1)\subset\Lambda^2V_1},\quad \mathbf{d(V_2)\subset\Lambda^2V_1+V_1\wedge V_2}.$$

For each $k\geq 0$ we define a filtration

$$\Lambda^{2,k-2}\subset\Lambda^{2,k-2}+\Lambda^{1,k-1}\subset\Lambda^{2,k-2}+\Lambda^{1,k-1}+\Lambda^{0,k}=\Lambda^k(\mathfrak{g}^*)$$

Taking the intersection with Z^k:

$$Z_2^k\subset Z_1^k\subset Z_0^k=Z^k,$$

and we have the groups: $\quad H_p^k=\dfrac{Z_p^k}{B^k\cap Z_p^k}.$

We define the cohomology groups: $\quad H^{p,q}(\mathfrak{g},V_1)=\dfrac{H_p^{p+q}}{H_{p+1}^{p+q}}.$

We consider the spaces:

$$D_j=\operatorname{ann}(\ker\{d\colon\Lambda^j\mathfrak{g}^*\to\Lambda^{j+1}\mathfrak{g}^*\}),\quad L_j^\phi=\operatorname{ann}(\ker\{\Lambda^j\mathfrak{g}^*\to\Lambda^{j+k}\mathfrak{g}^*,\quad \alpha\to\alpha\wedge\phi\}).$$

Proposition 2.1 *Let $\mathfrak{g}$ be a 5-dimensional Lie algebra. Then $\mathfrak{g}$ has a coherent splitting, with $\dim V_1=2$ and $H^{0,2}(\mathfrak{g},V_1)=0=H^{0,3}(\mathfrak{g},V_1)$, if and only if there exists a nonzero 2-form ϕ such that*

- $\phi\wedge\phi=0$;
- $d\phi=0$;
- $\mathcal{L}_X\phi$ *is a multiple of ϕ for all X in $\mathfrak{g}$;*
- $L_2^\phi\subset D_2$, $L_3^\phi\subset D_3$.

Theorem 2.2 *If $\mathfrak{g}$ has dimension 5, and there exists a coherent splitting $\mathfrak{g}^*=V_1\oplus V_2$ with $\dim V_1=2$ and $H^{0,3}(\mathfrak{g},V_1)=0=H^{0,2}(\mathfrak{g},V_1)$, then there is no hypo structure.*

For any 1-form γ we considerer: $\quad L_\gamma\colon\Lambda^j\to\Lambda^{j+1},\quad L_\gamma(\eta)=\gamma\wedge\eta.$

Proposition 2.3 *Let α,β be linearly independent one-forms on a Lie algebra $\mathfrak{g}$, and set $V=\ker L_\alpha\cap Z^3$. Then there is no hypo structure on $\mathfrak{g}$ of the form (α,ω_i) when either*

- $\dim L_\beta(V)<2$; *or*
- $\dim L_\beta(V)=2$ *and* $L_\alpha(L_\beta(Z^2))\subset L_\beta(V)$.

For **non-unimodular** Lie algebras, it turns out that we have a canonical choice for β:

Proposition 2.4 *Let $\mathfrak{g}$ be a non-unimodular Lie algebra and let $\beta\in\mathfrak{g}^*$ be the form corresponding to the linear map $\mathfrak{g}\to\mathbb{R}$, $X\to\operatorname{tr}\operatorname{ad}X$. If $\mathfrak{g}$ has a hypo structure (α,ω_i), then α and β are orthogonal with respect to the underlying metric.*

Thus, in the non-unimodular case Proposition 2.3 gives a fairly straightforward criterion:

Theorem 2.5 *Let $\mathfrak{g}$ be a* **non-unimodular** *Lie algebra, and let $\beta\in\mathfrak{g}^*$ be the form corresponding to the linear map $\mathfrak{g}\to\mathbb{R}$, $X\to\operatorname{tr}\operatorname{ad}X$. Suppose that either*

- $\dim L_\beta(Z^3)<2$; *or*
- $\dim L_\beta(Z^3)=2$ *and for every $\alpha\in\mathfrak{g}^*$ such that $L_\alpha(L_\beta(Z^3))=0\Rightarrow L_\alpha(L_\beta(Z^2))\subset L_\beta(Z^3)$.*

Then $\mathfrak{g}$ has no hypo structure.

3 Solvable Lie algebras with a hypo structure

Definition 3.1 *A contact form on a manifold M^{2n+1} is a differential 1-form μ such that $(d\mu)^n\wedge\mu\neq 0$ pointwise over M.*

Diatta gives a list of 24 (families of) indecomposable five-dimensional solvable Lie algebras $D_1,\dots,D_{24}$ that admit a left-invariant contact 1-form. They correspond to the algebras $A_{5,k}$ of [Mub] under

$$D_k=A_{5,k+3},\ k=1,2,3,\qquad D_k=A_{5,k+15},\ 4\leq k\leq 24.$$

We shall use the notation $D_k(p_1,\dots,p_n)$, $A_{5,k}(p_1,\dots,p_n)$ to denote special instances of a family for assigned values of the parameters.

Theorem 3.2 *A solvable Lie algebra of dimension five has a hypo structure if and only if it appears in following table*

Name	*Structure constants*	*Hypo structure*	μ
D_1 *[CS]*	$(e^{24}+e^{35},0,0,0,0)$	e^2,e^4,e^3,e^5,e^1	e^1
D_2 *[CS]*	$(e^{34}+e^{25},e^{35},0,0,0)$	$e^2,e^4,e^5,-e^1,-e^1+e^3$	e^5
$D_4(-\frac{1}{2},-\frac{3}{2})$	$(-\frac{1}{2}e^{15}-e^{23},-e^{25},\frac{1}{2}e^{35},\frac{3}{2}e^{45},0)$	e^1,e^3,e^2,e^5,e^4	e^1+e^4
$D_4(1,-3)$	$(-e^{23}-2e^{15},-e^{25},-e^{35},3e^{45},0)$	$e^5,\frac{1}{5}(e^1-e^4),\frac{1}{2}e^3,\frac{1}{5}e^2,-\frac{1}{15}(3e^1+e^4)$	e^1+e^4
$D_4(-2,3)$	$(e^{15}-e^{23},-e^{25},2e^{35},-3e^{45},0)$	$-e^3,2e^5,-2e^1,2e^2,-e^4\sqrt{2}$	e^1+e^4
$D_{15}(-1)$	$(-e^{15}-e^{24},-e^{34},e^{35},-e^{45},0)$	$\frac{1}{2}(-e^1+e^3),e^5,\frac{\sqrt{2}}{2}e^4,\frac{\sqrt{2}}{2}e^2,-e^1-e^3$	$e^1+e^2+e^3$
$D_{18}(-1,-1)$	$(-e^{14},-e^{25},e^{34}+e^{35},0,0)$	$-\frac{1}{2\sqrt{3}}(e^4+2e^5),\frac{1}{2\sqrt{3}}(e^2-e^3),$ $\frac{1}{3}e^1-\frac{1}{6}e^2-\frac{1}{6}e^3,\frac{1}{2}e^4,\frac{1}{3}(e^1+e^2+e^3)$	$e^1+e^2+e^3$
$D_{20}(-2,0)$	$(2e^{14},-e^{24}-e^{35},e^{25}-e^{34},0,0)$	$3e^2,-\sqrt{3}(3e^1+e^4+2e^5+2e^2+e^3),$ $9e^1+3e^3+3e^4,-\sqrt{3}(2e^4+e^2),$ $-5e^1-2e^2-4e^3-e^4+2e^5$	$e^1+e^2+e^3$
D_{22}	$(e^{23}+2e^{14},e^{24}+e^{35},e^{34}-e^{25},0,0)$	$e^4,e^1,\frac{\sqrt{2}}{2}e^3,\frac{\sqrt{2}}{2}e^2,\frac{1}{3}(e^5-3e^1)$	$e^1+e^2+e^3$
$A_{3,8}\oplus(0,12)$	$(e^{23},-e^{13},0,0,e^{45})$	e^1,e^2,e^4,e^3,e^5	$e^1+e^2+e^5$
$A_{5,1}$ *[CS]*	$(e^{35},e^{45},0,0,0)$	e^1,e^3,e^2,e^4,e^5	$--$
$A_{5,2}$ *[CS]*	$(e^{25},e^{35},e^{45},0,0)$	e^1,e^4,e^3,e^2,e^5	$--$
$A_{5,7}(p,-p,-1)$	$(e^{15},pe^{25},-pe^{35},-e^{45},0)$	e^1,e^4,e^2,e^3,e^5	$--$
$A_{5,8}(-1)$	$(e^{25},0,e^{35},-e^{45},0)$	e^1,e^2,e^3,e^4,e^5	$--$
$A_{5,13}(-1,0,r)$	$(e^{15},-e^{25},re^{45},-re^{35},0)$	e^1,e^2,e^3,e^4,e^5	$--$
$A_{5,14}(0)$	$(e^{25},0,e^{45},-e^{35},0)$	e^1,e^2,e^3,e^4,e^5	$--$
$A_{5,15}(-1)$	$(e^{15}+e^{25},e^{25},e^{45}-e^{35},-e^{45},0)$	e^1,e^4,e^3,e^2,e^5	$--$
$A_{5,17}(0,0,r)$	$(e^{25},-e^{15},re^{45},-e^{35}r,0)$	e^1,e^2,e^3,e^4,e^5	$--$
$A_{5,17}(p,-p,1)$	$(e^{25}+pe^{15},pe^{25}-e^{15},$ $e^{45}-pe^{35},-e^{35}-pe^{45},0)$	e^1,e^3,e^2,e^4,e^5	$--$
$A_{5,17}(p,-p,-1)$	$(e^{25}+pe^{15},pe^{25}-e^{15},$ $-e^{45}-pe^{35},e^{35}-pe^{45},0)$	e^1,e^3,e^4,e^2,e^5	$--$
$A_{5,18}(0)$	$(e^{35}+e^{25},-e^{15}+e^{45},e^{45},-e^{35},0)$	e^1,e^3,e^2,e^4,e^5	$--$
$A_{3,1}\oplus\mathbb{R}^2$ *[CS]*	$(0,0,0,0,0)$	e^1,e^2,e^4,e^5,e^3	$--$
$A_{3,3}\oplus\mathbb{R}^2$ *[CS]*	$(e^{23},0,0,0,0)$	e^1,e^2,e^4,e^5,e^3	$--$
$A_{3,6}\oplus\mathbb{R}^2$	$(e^{13},-e^{23},0,0,0)$	e^1,e^2,e^4,e^5,e^3	$--$
$A_{3,8}\oplus\mathbb{R}^2$	$(e^{23},-e^{13},0,0,0)$	e^1,e^2,e^4,e^5,e^3	$--$

[CS] nilpotent, *indecomposable*, *decomposable*

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*The results given here are included in the paper [CFS].